2.8b stability in first-order systems

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Recall: $x_{t+1} = f(x_t)$ is (locally asymptotically) stable at an equilibrium $\bar{x}$ if $|f'(\bar{x})| < 1$
unstable at an equilibrium $\bar{x}$ if $|f'(\bar{x})| > 1$.

What is the equivalent for first-order systems?

Thm: Let $X(t+1) = F(X(t))$ be a system of n first-order equations,

$X(t) = (x_1(t), \dots, x_n(t))^T$, $F = (f_1, \dots, f_n)^T$, and $f_i = f_i(x_1, \dots, x_n)$.

Let $\bar{X}$ be an equilibrium of the system. Then linearization of the system about $\bar{X}$ and letting $U(t) = X(t) - \bar{X}$ gives a system

$$U(t+1) = J\, U(t),$$

where J is the Jacobian matrix of F at $\bar{X}$,

$$J(\bar{X}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\bar{X}) & \frac{\partial f_1}{\partial x_2}(\bar{X}) & \cdots & \frac{\partial f_1}{\partial x_n}(\bar{X}) \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\bar{X}) & \cdots & \cdots & \frac{\partial f_n}{\partial x_n}(\bar{X}) \end{pmatrix}$$

Then $\bar{X}$ is locally asymptotically stable if $|\lambda_i| < 1$ $\forall$ eigenvalues λ_i

and unstable if some $|\lambda_i| > 1$.

Proof sketch: For $X(0)$ s.t. $|\bar{X} - X(0)| < \varepsilon$, with ε sufficiently small,

can approximate $X(t+1) = F(X(t))$ by the Taylor Series of F

$$X(t+1) \approx F(\bar{X}) + \underbrace{J(\bar{X})}_{\text{Jacobian}}(X(t) - \bar{X}) + \cancel{\tfrac{1}{2}(X(t)-\bar{X})^T \underbrace{H(\bar{X})}_{\text{Hessian}} (X(t)-\bar{X})} + \cdots$$

$$\Rightarrow X(t+1) - \bar{X} \approx J(\bar{X})(X(t) - \bar{X}) \quad \text{for } X(t) \text{ sufficiently close to } \bar{X}.$$

$$U(t+1) \approx J(\bar{X})\, U(t)$$

$$\Rightarrow U(t) = [J(\bar{X})]^t\, U(0).$$

If all eigenvalues $|\lambda_i| < 1$, then $\rho(J(\bar{X})) < 1$.

If $\rho(J(\bar{X})) < 1$, then $\lim_{t\to\infty} [J(\bar{X})]^t \to 0$, so $\lim_{t\to\infty} X(t) = \bar{X}$.

If $|\lambda_i| > 1$ for some i, then so long as $[X(0) - \bar{X}] \cdot V_i \neq 0$, where V_i is the eigenvector associated with $|\lambda_i| > 1$, then $\lim_{t\to\infty} |U(t)| = \infty$.

Of course, the linearization may break down as $|X(t) - \bar{X}|$ grows, but $X(t)$ will still leave a sufficiently small ball around $\bar{X}$, so it is unstable. ▱

Thm 2.10 Let $J \in \mathbb{R}^{2\times 2}$. Then $|\lambda_i| < 1$ $\forall$ eigenvalues λ_i iff

$$|Tr(J)| < 1 + \det(J) < 2.$$

And $|\lambda_i| > 1$ for some eigenvalue λ_i if at least one of the following is true:

$$Tr(J) > 1 + \det(J), \quad Tr(J) < -1 - \det(J), \quad \det(J) > 1.$$

proof. Let $\tau = Tr(J)$ and $\delta = \det(J)$.

The characteristic eqn of a 2×2 matrix is

$$p(\lambda) = \lambda^2 - \tau\lambda + \delta = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$$

Forward Case 1: $|\lambda_i| < 1$, $\lambda_{1,2} \in \mathbb{R}$

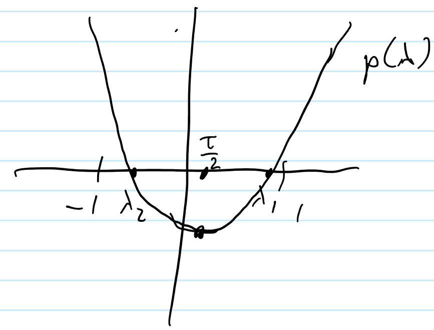
$\Rightarrow \tau^2 \geq 4\delta$ [illegible]

Forward Case 1: $|\lambda_i|<1$, $\lambda_{1,2}\in\mathbb{R}$

$$\Rightarrow \tau^2 \ge 4\delta. \quad \text{Also, } \tau = \lambda_1+\lambda_2, \quad |\tau| \le |\lambda_1| + |\lambda_2| < 2.$$

Furthermore, the function $p(\lambda)$ is a parabola with vertex at $\frac{\tau}{2}$, and zeros at λ_1 and λ_2. WLOG, $\lambda_2 < \lambda_1$.

Therefore, $-1 < \lambda_2 \le \frac{\tau}{2} \le \lambda_1 < 1$



$$\Rightarrow \frac{|\tau|}{2} < 1, \quad \text{so} \quad 4 > \tau^2 \ge 4\delta \Rightarrow \delta < 1.$$

Also, $\left|\frac{\tau}{2}-1\right| > \left|\frac{\tau}{2}-\lambda_1\right|$ and $\left|\frac{\tau}{2}+1\right| > \left|\frac{\tau}{2}-\lambda_2\right|$

Recall $|\tau|<2$, so $\frac{\tau}{2}-1<0$.

$$\Rightarrow 1-\frac{\tau}{2} > \left|\frac{\tau}{2}-\lambda_1\right| \quad \text{and} \quad 1+\frac{\tau}{2} > \left|\frac{\tau}{2}-\lambda_2\right|$$

$$\Rightarrow 1 \pm \frac{\tau}{2} > \left|\frac{\tau}{2}-\lambda_i\right|$$

But $\left|\frac{\tau}{2}-\lambda_i\right| = \frac{\sqrt{\tau^2-4\delta}}{2}$, so

$$1 - \frac{|\tau|}{2} > \frac{\sqrt{\tau^2-4\delta}}{2}$$

$$\Rightarrow 1 - |\tau| + \frac{\tau^2}{4} > \frac{\tau^2-4\delta}{4}$$

$$\Rightarrow 1 - |\tau| > -\delta$$

$$\Rightarrow 1+\delta > |\tau|$$

$$\Rightarrow |\tau| < 1+\delta.$$

Furthermore, $\delta = \lambda_1\lambda_2$, so $|\delta|<1 \Rightarrow 1+\delta < 2$

Forward case 2: $|\lambda_i|<1$ and $\bar{\lambda}_1 = \lambda_2$ are complex conjugates.

Then $\tau^2 < 4\delta$

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$$\lambda_{1,2} = \frac{\tau}{2} \pm i \cdot \frac{\sqrt{4\delta - \tau^2}}{2}$$

$$\Rightarrow |\lambda_i|^2 = \frac{\tau^2}{4} + \delta - \frac{\tau^2}{4} = \delta.$$

$$\Rightarrow 0 < \delta < 1.$$

But $|\tau| < 2\sqrt{\delta} \leq 1 + \delta$ $\quad (a^2 + b^2 \geq 2ab)$

$$\Rightarrow |\tau| < 1 + \delta < 2.$$

Backward case 1: $|\tau| < 1 + \delta < 2$ and $\lambda_i \in \mathbb{R}$, so $\tau^2 \geq 4\delta$.

Let $\lambda_1 = \frac{\tau}{2} + \frac{\sqrt{\tau^2 - 4\delta}}{2}$, $\lambda_2 = \frac{\tau}{2} - \frac{\sqrt{\tau^2 - 4\delta}}{2}$, so $\lambda_2 \leq \lambda_1$.

Note $|\tau| < 1 + \delta$

$$\Rightarrow 1 - |\tau| > -\delta$$

$$\Rightarrow \left(1 - \frac{|\tau|}{2}\right)^2 > \frac{\tau^2}{4} - \delta \geq 0$$

$$\Rightarrow 1 - \frac{|\tau|}{2} > \frac{\sqrt{\tau^2 - 4\delta}}{2}$$

$$\Rightarrow 1 > \frac{|\tau|}{2} + \frac{\sqrt{\tau^2 - 4\delta}}{2} \geq \lambda_1$$

and $-1 < \frac{-|\tau|}{2} - \frac{\sqrt{\tau^2 - 4\delta}}{2} \leq \lambda_2$, so $-1 < \lambda_2 \leq \lambda_1 < 1$.

Backward case 2: $|\tau| < 1 + \delta < 2$ and λ_i are complex conjugates.

Then $|\lambda_i|^2 = \delta < 1$.

Thm 2.11 (Jury conditions, Schur-Cohn criterion, $n=3$)

Suppose $p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$, $a_1, a_2, a_3 \in \mathbb{R}$ is a polynomial

Then the solutions $\lambda_1, \lambda_2, \lambda_3$ of $p(\lambda)=0$ satisfy $|\lambda_i|<1$ iff

(1) $p(1) = 1 + a_1 + a_2 + a_3 > 0$

(2) $(-1)^3 p(-1) = 1 - a_1 + a_2 - a_3 > 0$,

(3) $1 - (a_3)^2 > |a_2 - a_3 a_1|$

(Necessary + sufficient conditions for $n=3$)

Thm 2.12 If the solutions $\lambda_1, \dots, \lambda_n$ of $p(\lambda)=0$, where

$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n$ satisfy $|\lambda_i|<1$, then

(1) $p(1) = 1 + a_1 + a_2 + \dots + a_n > 0$

(2) $(-1)^n p(-1) = 1 - a_1 + a_2 - \dots + (-1)^n a_n > 0$

(3) $|a_n| < 1$.

(Necessary but not sufficient)

Def. 2.9 Let $\bar{X}$ be an equilibrium of $X(t+1) = F(X(t))$

and let $J(\bar{X})$ be the Jacobian matrix at $\bar{X}$. $\bar{X}$ is

hyperbolic if $|\lambda_i| \neq 1$ $\forall$ eigenvalues λ_i of $J(\bar{X})$.

Otherwise, it is **nonhyperbolic**.

Ex. 2.15: Consider a predator-prey system

(prey) $x_{t+1} = x_t(a - x_t - y_t)$, $a > 0$

(predator) $y_{t+1} = y_t(b + x_t)$, $0 < b < 1$

Equilibria are $(0,0)$, $(a-1,0)$, and $(\bar{x},\bar{y}) = (1-b,\ a+b-2)$

$$J = \begin{pmatrix} \frac{\partial}{\partial x}\left[x(a-x-y)\right] & \frac{\partial}{\partial y}\left[x(a-x-y)\right] \\ \frac{\partial}{\partial x}\left[y(b+x)\right] & \frac{\partial}{\partial y}\left[y(b+x)\right] \end{pmatrix}$$

$$J = \begin{pmatrix} a-2x-y & -x \\ y & b+x \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$\lambda_{1,2} = a, b$

Since $0<b<1$, $(0,0)$ is locally asymptotically stable if $a<1$.

Intuitively, everything dies.

$$J(a-1,0) = \begin{pmatrix} 2-a & 1-a \\ 0 & a+b-1 \end{pmatrix}$$

$\lambda_{1,2} = 2-a,\ a+b-1$

Need $1<a<2-b$ to be locally asymptotically stable and nonnegative.

prey survives, but predator goes extinct

$$J(\bar{x},\bar{y}) = \begin{pmatrix} b & -\bar{x} \\ \bar{y} & 1 \end{pmatrix}$$

After some algebra using the Jury conditions for $n=2$, find $2<a+b<3$ to be locally asymptotically stable and nonnegative

prey survives, and predator survives.